Equivalent Continuum Representation of Structures Composed of Repeated Elements

John O. Dow,* Z. W. Su,† and C. C. Feng‡
University of Colorado, Boulder, Colorado
and
Carl Bodley§
Martin-Marietta Aerospace, Denver, Colorado

A procedure for determining the equivalent continuum properties of a structure composed of repeated patterns of discrete elements with both displacement and rotational coordinates is presented. These nodal coordinates are transformed to rigid-body and strain gradient variables using a polynomial approximation. The maximum number of independent variables that may be retained is determined by applying a ranking procedure to the resulting transformation matrix. The possibility of introducing errors by requiring the analyst to supply the strain gradient terms directly is reduced by identifying the appropriate variables through the use of the polynomial expansion and the ranking procedure. Additional constraints may be imposed in this analysis. The equivalent continuum parameters result when a further transformation to the appropriate kinematic variables is applied and the strain energy expression is reduced to these variables. Three-dimensional beam- and plate-like structures are treated. The results correspond to findings using other approaches.

Nomenclature

a_i, b_i, c_i	= polynomial coefficients
EA	= equivalent longitudinal rigidity
EI_y, EI_z	= equivalent flexural rigidity
GJ_x	= equivalent torsional rigidity
k_1GA, k_2GA	= equivalent shearing rigidity
$[K]$	= stiffness matrix of a finite element
$[K^*]$	= equivalent stiffness matrix
$\underline{p}, \underline{q}, \underline{r}$	= rotational components at any point of an element
p, q, r	= rotational components at the origin of an element
$[S]$	= stiffness matrix associating with $[\alpha\ \beta]$ vector
$[T_i]$	= coordinate transformation matrices
U	= strain energy of a finite element
U^*	= equivalent strain energy
$\underline{u}, \underline{v}, \underline{w}$	= displacement components at any point of an element
u, v, w	= displacement components at the origin of an element
$[u]_i$,	= nodal displacement vector
$[u]$	= strain gradient vector associating with $[T_1]$
x, y, z	= global coordinates
x_c, y_c, z_c	= global coordinates of the origin of an element
$[\alpha]$	= vector of kinematic parameters associating with $[T_3]$
$[\beta]$	= strain gradient vector associating with $[T_3]$
ϵ, γ	= strains at the origin of an element
κ	= equivalent continuum curvature terms
$[\mu]$	= reduced strain gradient vector associating with $[T_2]$
ξ, η, ζ	= local spatial variables

Presented as Paper 83-1007 at the AIAA/ASME/ASCE/AHS 24th Structures, Structural Dynamics and Materials Conference, Lake Tahoe, NV, May 2-4, 1983; received March 27, 1984; revision received Oct. 30, 1984.

*Assistant Professor, Department of Civil Engineering.
†Visiting Scholar, from Shandong Mining College, China.
‡Professor, Department of Civil Engineering.
§Senior Engineer.

I. Introduction

THIS presentation develops a procedure for formulating an equivalent continuum representation of large complex beam- and plate-like structures composed of repeated elements. The objective of this procedure is to develop stiffness relationships for the repeated structural element or cell that are equivalent to the stiffness properties of a continuous beam or plate. For example, in the case of a beam-like structure, an equivalent flexural rigidity (*EI*) is sought for the repeated structural element. This *EI* can then be used to approximate the original beam-like problem in a differential equation or finite element model of a continuous beam.

The advantages of a continuum representation of a structure composed of repeated elements and several methods for formulating the continuum representation have been discussed in the literature.[1-18] The most significant method appears to be the approach developed by Noor and his colleagues in which the strain energy is represented in terms of strain gradient quantities.[19-24]

The method developed here also utilizes strain gradient variables but the analysis is not required a priori to specify the terms that a particular repeated element is capable of modeling. This set of linearly independent terms is identified by checking dependency relations between the strain gradient terms introduced into the analysis. In this way, an appropriate set of strain gradient terms is utilized and complex structures present no more difficulties than do simple structures.

In the approach presented here, the analyst is required to supply the stiffness matrix of a repeated element and to indicate the type of continuum that is being modeled. By checking the strain gradient relations for dependency relations, the procedure will produce the equivalent stiffness properties with the necessary kinematic constraints. However, if the analyst wishes to include additional constraints (such as requiring plane section to remain plane sections), they can easily be included. The development of the equivalent continuum representation proceeds as follows.

The equivalent stiffness relationships are developed by computing an energy expression for the repeated element in terms of the same independent variables used to express the energy density for the continuum model. These energy expressions are compared term by term and the equivalent coefficients are equated to one another. For example, in the case of a simple beam-like structure, an expression for the strain energy of the repeated cell is readily available in terms of displacements and rotations at the nodes from the finite element model. However, the kernel of strain energy density for a simple prismatic beam is $\frac{1}{2}EI\kappa^2$, where EI is the flexural rigidity and κ the curvature. By transforming the finite element energy expression for the repeated element so that it is expressed in terms of curvature, an equivalent flexural rigidity is produced. In physical terms, this process approximates the differential volume of the continuum with the repeated element, which is small with respect to the total structure.

In summary, the procedure consists of the following five steps:

1) A repeated structural cell is isolated.

2) A finite element model of the isolated cell is developed in terms of nodal displacement and rotation quantities.

3) A transformation matrix is constructed that relates the nodal quantities to kinematic quantities used to express the strain energy density of the continuum.

4) A strain energy expression is formulated for the repeated element in terms of the new coordinate variables. This expression is compared to the strain energy density for the continuum to determine the equivalent stiffness properties of the element.

5) These equivalent stiffness properties are used to approximate the original problem in a differential equation or a finite element solution.

Steps 1, 2, 4, and 5 are self-explanatory and will be illustrated with examples that follow. The overall transformation developed in the third step is composed of four successive transformations that combine to produce the desired result as follows:

1) A transformation $[T_1]$ that depends on geometry is generated, relating the displacements and rotations of the finite element model to rigid-body motion, strain, and strain gradient terms specified at the point chosen as the origin of the cell.

2) A transformation $[T_2]$ that eliminates linearly dependent strain gradient terms or incorporates the kinematic assumptions used in developing the governing differential equation of the continuum is used to eliminate specific strain and strain gradient terms introduced in step 1.

3) A transformation $[T_3]$ relating the strain and strain gradient terms retained in step 2 to the independent kinematic variables that define the strain energy in the appropriate continuum model is formulated.

4) A transformation $[T_4]$ similar to the static condensation procedure is used to reduce the strain energy expression to the desired variables.

II. Preliminary Development

The procedure for determining the nodal displacements, rotations, and strains in terms of a polynomial expansion whose coefficients depend on the rigid-body motion, strain, and strain gradient quantities at the origin of the local coordinate system will now be developed in detail. The development begins by expressing the displacement field (u, v, and w) as arbitrary polynomials around a local origin for the cell. The local spatial variables are defined in terms of the global coordinate system as

$$\xi = x - x_c; \quad \eta = y - y_c; \quad \zeta = z - z_c \tag{1}$$

where x_c, y_c, and z_c are the coordinates of the local origin (usually inside of the cell) with respect to the global coordinates x, y, and z of a general point.

For this presentation, the displacements are expressed as complete third-order polynomials. This order of representation was chosen because it contains far more terms than are required to represent the structures used in the examples. This allows these developments to be used directly on more complex structures and allows the procedure for identifying redundant linearly dependent strain gradient terms to be demonstrated. Furthermore, this order of representation produces the compatibility equations of elasticity as a by-product of the development. The third-order polynomials representing the displacements in the x, y, and z directions are

$$\underline{u}(\xi,\eta,\zeta) = a_1 + a_2\xi + a_3\eta + a_4\zeta + \ldots + a_{20}\zeta^3$$

$$\underline{v}(\xi,\eta,\zeta) = b_1 + b_2\xi + b_3\eta + b_4\zeta + \ldots + b_{20}\zeta^3$$

$$\underline{w}(\xi,\eta,\zeta) = c_1 + c_2\xi + c_3\eta + c_4\zeta + \ldots + c_{20}\zeta^3 \tag{2}$$

The 60 coefficients will be evaluated in terms of the quantities at the origin of the local coordinate system. The three coefficients of the zeroth-order terms can be evaluated immediately in terms of the rigid-body displacements at the origin as

$$u = a_1; \quad v = b_1; \quad w = c_1 \tag{3}$$

Table 1 Expressions for displacement, rotations, and strains

$a_1 = u$	$b_1 = v$	$c_1 = w$
$a_2 = \epsilon_x$	$b_2 = \gamma_{xy}/2 + r$	$c_2 = \gamma_{xz}/2 - q$
$a_3 = \gamma_{xy}/2 - r$	$b_3 = \epsilon_y$	$c_3 = \gamma_{yz}/2 + p$
$a_4 = q + \gamma_{xz}/2$	$b_4 = \gamma_{yz}/2 - p$	$c_4 = \epsilon_z$
$a_5 = \epsilon_{x,x}/2$	$b_5 = (\gamma_{xy,x} - \epsilon_{x,y})/2$	$c_5 = (\gamma_{xz,x} - \epsilon_{x,z})/2$
$a_6 = \epsilon_{x,y}$	$b_6 = \epsilon_{y,x}$	$c_6 = (-\gamma_{xy,z} + \gamma_{yz,x} + \gamma_{xz,y})/2$
$a_7 = \epsilon_{x,z}$	$b_7 = (\gamma_{xy,z} + \gamma_{yz,x} - \gamma_{xz,y})/2$	$c_7 = \epsilon_{z,x}$
$a_8 = (\gamma_{xy,y} - \epsilon_{y,x})/2$	$b_8 = \epsilon_{y,y}/2$	$c_8 = (\gamma_{yz,y} - \epsilon_{y,z})/2$
$a_9 = (\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y})/2$	$b_9 = \epsilon_{y,z}$	$c_9 = \epsilon_{z,y}$
$a_{10} = (\gamma_{xz,z} - \epsilon_{z,x})/2$	$b_{10} = (\gamma_{yz,z} - \epsilon_{z,y})/2$	$c_{10} = \epsilon_{z,z}/2$
$a_{11} = \epsilon_{x,xx}/6$	$b_{11} = (\gamma_{xy,xx} - \epsilon_{x,xy})/6$	$c_{11} = (\gamma_{xz,xx} - \epsilon_{x,xz})/6$
$a_{12} = \epsilon_{x,xy}/2$	$b_{12} = \epsilon_{y,xx}/2$	$c_{12} = (\gamma_{xz,xy} - \epsilon_{x,yz})/2$
$a_{13} = \epsilon_{x,xz}/2$	$b_{13} = (\gamma_{xy,xz} - \epsilon_{x,yz})/2$	$c_{13} = \epsilon_{z,xx}/2$
$a_{14} = (\gamma_{xy,yy} - \epsilon_{y,xy})/6$	$b_{14} = \epsilon_{y,yy}/6$	$c_{14} = (\gamma_{yz,yy} - \epsilon_{y,yz})/6$
$a_{15} = \epsilon_{x,yy}/2$	$b_{15} = \epsilon_{y,xy}/2$	$c_{15} = (\gamma_{yz,xy} - \epsilon_{y,xz})/2$
$a_{16} = (\gamma_{xy,yz} - \epsilon_{y,xz})/2$	$b_{16} = \epsilon_{y,yz}/2$	$c_{16} = \epsilon_{z,yy}/2$
$a_{17} = (\gamma_{xz,zz} - \epsilon_{z,xz})/6$	$b_{17} = (\gamma_{yz,zz} - \epsilon_{z,yz})/6$	$c_{17} = \epsilon_{z,zz}/6$
$a_{18} = \epsilon_{x,zz}/2$	$b_{18} = (\gamma_{yz,xz} - \epsilon_{z,xy})/2$	$c_{18} = \epsilon_{z,xz}/2$
$a_{19} = (\gamma_{xz,yz} - \epsilon_{z,xy})/2$	$b_{19} = \epsilon_{y,zz}/2$	$c_{19} = \epsilon_{z,yz}/2$
$a_{20} = \epsilon_{x,yz}$	$b_{20} = \epsilon_{y,xz}$	$c_{20} = \epsilon_{z,xy}$

The coefficients of the first-order terms are evaluated as a function of rotations and strains. The rotations will be considered first. The small displacement approximation for the rotations $\underline{p}$, $\underline{q}$, and $\underline{r}$, around the x, y, and z axes, respectively, of a volume element is given as

$$\begin{Bmatrix} \underline{p} \\ \underline{q} \\ \underline{r} \end{Bmatrix} = \tfrac{1}{2}\,\mathrm{curl} \begin{Bmatrix} \underline{u} \\ \underline{v} \\ \underline{w} \end{Bmatrix} \tag{4}$$

When the series expansions for the displacements are differentiated, substituted into the expressions for the rotations, and evaluated at the origin of the local coordinate system, the results are

$$p = \tfrac{1}{2}(c_3 - b_4); \quad q = \tfrac{1}{2}(a_4 - c_2); \quad r = \tfrac{1}{2}(b_2 - a_3) \tag{5}$$

The strain relations in terms of small displacements are

$$\epsilon_x = \underline{u}_{,x}; \quad \epsilon_y = \underline{v}_{,y}; \quad \epsilon_z = \underline{w}_{,z}$$

$$\gamma_{xy} = \underline{u}_{,y} + \underline{v}_{,x}; \quad \gamma_{xz} = \underline{u}_{,z} + \underline{w}_{,x}; \quad \gamma_{yz} = \underline{v}_{,z} + \underline{w}_{,y} \tag{6}$$

where the subscripts following a comma indicate that derivatives have been taken. When the series expansion for the displacements [Eq. (2)] are differentiated, substituted into the strain expressions, and evaluated at the origin of the local system, the results are

$$\epsilon_x = a_2; \quad \epsilon_y = b_3; \quad \epsilon_z = c_4$$

$$\gamma_{xy} = b_2 + a_3; \quad \gamma_{xz} = c_2 + a_4; \quad \gamma_{yz} = c_3 + b_4 \tag{7}$$

The nine coefficients of the first-order terms, $a_2, a_3 \ldots c_4$, evaluated in terms of the rigid-body rotations and the strains at the origin, are shown in Table 1.

The coefficients of the second-order terms are evaluated in terms of the first derivatives of the strains. Since there are 6 strains and 3 spatial variables, 18 first derivative terms exist. The 18 equations created when the first derivatives of the strains are taken and evaluated at the local origin allow the 18 coefficients of the second-order terms $a_5, b_5 \ldots c_{10}$ to be evaluated. These terms are shown in Table 1.

The 30 coefficients of the third-order terms are evaluated as functions of the second derivatives of the strains. Since there are 6 strains and 6 second derivatives, 36 second derivative strain gradient terms exist. This means that 36 equations are available to evaluate the 30 remaining coefficients. It is found that the six compatibility equations of elasticity result from this overdetermined set of equations. The results of evaluating the remaining 30 terms—a_{11}, b_{11}, and c_{20}—are given in Table 1. The displacements, rotations, and strains over the region can be expressed using the coefficients evaluated in Table 1. The quantities developed will be used to formulate the transformations $[T_1]$, $[T_2]$, and $[T_3]$.

III. Transformation $[T_1]$ from Global Finite Element Nodal Coordinates to Local Strain and Gradient Variables

The first step in formulating the desired equivalent continuum representation of the total structure is the transformation of the strain energy expression of the cell from standard finite element coordinates to rigid-body, strain, and strain gradient variables. This is accomplished by applying the polynomial expansion of the displacements and rotations developed in the previous section to each of the displacement and rotation coordinates of the finite element model. The formulation of the transformation matrix $[T_1]$ can be illustrated by a two-dimensional case having six nodes. This 6 node model has 12 displacement and 6 rotation coordinates for a total of 18 degrees of freedom. Thus, the overall transformation matrix $[T_1]$ for this case will have 18 rows and the 60 columns shown in Table 1. For this two-dimensional case, $[T_1]$ is the 18×60 matrix

$$[u]_i = \begin{bmatrix} u_1 \\ v_1 \\ r_1 \\ u_2 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_i = \begin{bmatrix} 1 & 0 & 0 & \zeta_1 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \eta_1 & \cdot \\ 0 & 0 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 0 & \zeta_2 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \eta_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u \\ v \\ r \\ \epsilon_x \\ \epsilon_y \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = [T_1][u] \tag{8}$$

It is noted that this transformation matrix depends only on the geometry of the cell and the location of the local origin. In general, it will be a rectangular matrix in which the number of rows depends on the number of degrees of freedom in the finite element model of the original cell.

The strain energy for a repeated element can be expressed in strain and strain gradient coordinates as follows:

$$U = (\tfrac{1}{2})[u]_i^T[K][u]_i = (\tfrac{1}{2})[u]^T[T_1]^T[K][T_1][u] \tag{9}$$

where $[K]$ is the finite element stiffness matrix for the repeated element.

The procedure illustrated here for a two-dimensional beam-like structure is applicable to other cases. Thus, the transformation matrix $[T_1]$ for two- or three-dimensional beam- and plate-like structures is a function of only the geometry of the problem.

IV. Reduction Transformation $[T_2]$

All of the 60 variables contained in Table 1 are not retained in most analyses because of dependency relations resulting from the configuration of the structure or due to kinematic constraints imposed on the problem. Since the rigid-body terms do not contribute to the strain energy, they can be discarded at the beginning of the analysis. The rows and columns associated with these quantities are eliminated in order to reduce the size of the problem.

The reduction transformation is used to eliminate linearly dependent strain gradient terms or to incorporate the kinematic assumptions used in developing the governing differential equation for the equivalent continuum problem. Dependency relations between various strain gradient terms obviously exist when the stiffness matrix being transformed has less than 60 degrees of freedom. A complete description of the possible deformations of a structure may not require all of the strain gradient terms in a specific direction.

A simple example of such a deformation is the extensional mode of the structure shown in Fig. 1. The terms ϵ_x and $\epsilon_{x,x}$ can both be represented in the structure because there are two constant strain bars in the x direction. Both bars can have either the same strain or each bar can have different strains. These two states allow ϵ_x and $\epsilon_{x,x}$ to be represented. However, $\epsilon_{x,xx}$ cannot be modeled independently because there is no third strain state possible with only two bars.

Additional strain gradient terms may be eliminated by the imposition of kinematic constraints. For example, the Love-Kirchhoff approximations that straight lines normal to the middle surface before deformation remain straight, normal to the middle surface, and unchanged in length after deformation could be stated in terms of displacements and strains as follows:

1) All of the strain gradient terms in the v and w directions that are functions of y and z will be neglected. This assumes the shape of the sections to remain unchanged.

2) All of the strain gradient terms in the u direction that are not linear functions of y and z and functions of x will be neglected. The terms retained are x,y,z, xy, xz, x^2y, and x^2z. This constrains the planes of the cross section to remain planes.

3) The shear strains γ_{xy} and γ_{xz} and their derivatives are neglected. This forces the planes normal to the middle surface to remain normal.

The constraints just discussed will now be used to reduce the number of strain gradient variables associated with the repeated element shown in Fig. 1. The variables eliminated by the Love-Kirchhoff approximation shown by * in Table 2 are determined by inspecting the expression for the displacements given in Table 1. The original 60 variables (54 strain gradient and 6 rigid-body terms) are reduced to 7 strain gradient variables. However, the previous discussion concerning $\epsilon_{x,xx}$ has shown that this term must be neglected for a structure with two bar elements in the x direction. Hence $\epsilon_{x,xx}$ must be eliminated for this structure. The artificial constraints imposed by the Love-Kirchhoff approximations and the one natural constraint on $\epsilon_{x,xx}$ have reduced the possible number of variables to six. Thus the final transformation $[T_2]$ will be of order 60×6. A more general formulation technique for $[T_2]$ will now be developed.

The repeated element in Fig. 1 has 21 flexible degrees of freedom. This means that 33 of the 54 strain gradient variables are redundant. The linearly dependent strain gradient variables can be determined with an elimination procedure that determines the rank of the transformation $[T_1]$.

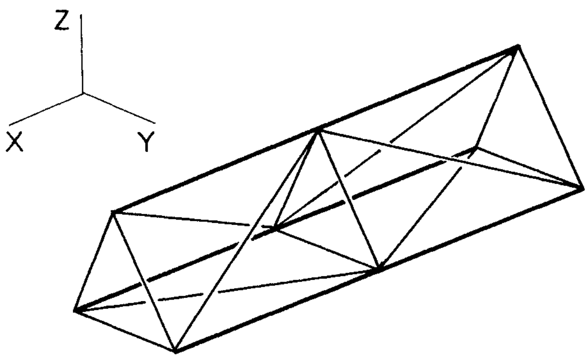

Fig. 1 Repeated element of a double-bay single-laced beam-like structure.

Table 2 Sixty terms in a third-order polynomial expansion of displacements in three dimensions

$\rightarrow u_0$	$\rightarrow v_0$	$\rightarrow w_0$	$\epsilon_{x,xz}$	$^{+*}\epsilon_{y,xz}$	$^{+*}\epsilon_{z,xz}$
$\rightarrow p_0$	$\rightarrow q_0$	$\rightarrow r_0$	$^{+*}\epsilon_{x,yz}$	$^{+*}\epsilon_{y,yz}$	$^{+*}\epsilon_{z,yz}$
ϵ_x	$^{*}\epsilon_y$	$^{*}\epsilon_z$	$^{*}\gamma_{xy}$	$^{*}\gamma_{xz}$	$^{*}\gamma_{yz}$
$\epsilon_{x,x}$	$^{\#*}\epsilon_{y,x}$	$^{\#*}\epsilon_{z,x}$	$^{*}\gamma_{xy,x}$	$^{*}\gamma_{xz,x}$	$^{\#*}\gamma_{yz,x}$
$\epsilon_{x,y}$	$^{+*}\epsilon_{y,y}$	$^{+*}\epsilon_{z,y}$	$^{+*}\gamma_{xy,y}$	$^{+*}\gamma_{xz,y}$	$^{+*}\gamma_{yz,y}$
$\epsilon_{x,z}$	$^{+*}\epsilon_{y,z}$	$^{+*}\epsilon_{z,z}$	$^{*}\gamma_{xy,z}$	$^{+*}\gamma_{xz,z}$	$^{+*}\gamma_{yz,z}$
$^{+}\epsilon_{x,xx}$	$^{*}\epsilon_{y,xx}$	$^{*}\epsilon_{z,xx}$	$^{+*}\gamma_{xy,xx}$	$^{+*}\gamma_{xz,xx}$	$^{+*}\gamma_{yz,yy}$
$^{+*}\epsilon_{x,yy}$	$^{+*}\epsilon_{y,yy}$	$^{+*}\epsilon_{z,yy}$	$^{+*}\gamma_{xy,yy}$	$^{+*}\gamma_{xz,zz}$	$^{+*}\gamma_{yz,zz}$
$^{+*}\epsilon_{x,zz}$	$^{+*}\epsilon_{y,zz}$	$^{+*}\epsilon_{z,zz}$	$^{*}\gamma_{xy,xz}$	$^{*}\gamma_{xz,xy}$	$^{+*}\gamma_{yz,xy}$
$\epsilon_{x,xy}$	$^{+*}\epsilon_{y,xy}$	$^{+*}\epsilon_{z,xy}$	$^{+*}\gamma_{xy,yz}$	$^{+*}\gamma_{xz,yz}$	$^{+*}\gamma_{yz,xz}$

→ Rigid-body displacements eliminated.

*Rigid-body displacement terms eliminated. Strain gradient terms eliminated by the Love-Kirchhoff assumptions.

+ Strain gradient terms eliminated due to structural configuration and element characteristics.

Strain gradient terms eliminated due to compatibility requirements at repeated element interface.

When this procedure is applied to $[T_1]$, all of the terms marked with a + in Table 2 are eliminated. In this case, the transformation $[T_2]$ is of order 60×21. Let $[\mu]$ be the vector containing the reduced 21 strain gradient variables, then the general term $[u]$ in Eq. (8) is transformed by

$$[u] = [T_2][\mu] \tag{10}$$

The constraints identified by the ranking procedure can be given the following interpretation:

1) The three displacement components u, v, and w have a linear variation in the plane of the cross section. Therefore, the coefficients of the quadratic and cubic y and z terms are eliminated. The higher-order terms in x are retained.

2) The configuration of this structure cannot model the terms $\gamma_{xy,y}$, $\gamma_{xz,z}$, $\epsilon_{x,xx}$, $\gamma_{xy,xx}$, and $\gamma_{xz,xx}$ because of lack of elements. The term $x_{yz,z}$ is a linear function of $\gamma_{xy,z}$ and $\gamma_{xz,y}$.

Any additional kinematic constraints can be imposed by eliminating strain gradient terms after the ranking operation. Such a step is necessary to eliminate any strain gradient terms that would cause adjoint sections to violate compatibility between repeated elements. For the case being discussed, the following three strain gradient terms must be eliminated: $\epsilon_{y,x}$, $\epsilon_{z,x}$, and $\gamma_{yz,x}$. They are noted in Table 2 by a #.

The further reduction of the original repeated element to an equivalent continuum model will be obtained by employing the transformations $[T_3]$ and $[T_4]$ to be discussed in Sec. V. This approach has been applied to the example problems discussed in Sec. VI.

V. Reduction to Continuum Variables

The desired equivalent continuum representation is obtained by transforming the strain energy expression to kinematic coordinates. These coordinates are the independent variables contained in the strain energy expression for the continuum model. The strain energy for the continuum equivalent of the beam-like structure shown in Fig. 1 can be written as

$$U = \frac{1}{2}\int_0^{\ell} (GJ_x\kappa_1^2 + EI_y\kappa_2^2 + EI_z\kappa_z^2 + EA\epsilon_x^2 + k_1GA\gamma_{xy}^2 + k_2GA\gamma_{xz}^2)\,dx \tag{11}$$

where

$$\kappa_1 = \tfrac{1}{2}(\gamma_{xz,y} - \gamma_{xy,z}); \quad \kappa_2 = -\epsilon_{x,y}; \quad \kappa_3 = \epsilon_{x,z}$$

The strain gradient terms in $[\mu]$ in Eq. (10) can be combined and rearranged into two groups through the transformation matrix $[T_3]$

$$[\mu] = [T_3]\left[{\alpha \atop \beta}\right] \tag{12}$$

in which the submatrix $[\alpha]$ contains the kinematic coordinates contained in Eq. (11) and the second submatrix $[\beta]$ contains the remaining strain gradient terms not directly included in the kinematic coordinates. The later variables will be condensed by use of transformation $[T_4]$.

For the example being considered, a transformation including all of the 6 kinematic variables contained in Eq. (11) is possible for the case with 18 strain gradient terms (the terms in Table 2 not eliminated by a +, →, or #). When the definitions of the kinematic variables in Eq. (11) are compared to the strain gradient terms retained in Table 2, it is seen that all of the required terms are present. However, when the same repeated element is constrained by the Love-Kirchhoff approximations, all six of the kinematic variables cannot be modeled because no shear terms are retained.

After the transformation $[T_3]$, the strain energy is expressed in terms of the required kinematic quantities and the remaining strain gradients as

$$U = \tfrac{1}{2}[u]^T[T_1]^T[K][T_1][u]$$
$$= \tfrac{1}{2}[{}^{\alpha}_{\beta}]^T[T_3]^T[T_2]^T[T_1]^T[K][T_1][T_2][T_3][{}^{\alpha}_{\beta}]$$
$$= \tfrac{1}{2}[{}^{\alpha}_{\beta}]^T[S][{}^{\alpha}_{\beta}] \tag{13}$$

where

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$
$$= [T_3]^T[T_2]^T[T_1]^T[K][T_1][T_2][T_3] \tag{14}$$

is the result of successive transformations associating with the original stiffness matrix.

The final step in developing the equivalent continuum representation is to reduce the strain energy expression to a function of the kinematic variables alone by applying a reduction analogous to a static condensation procedure. This condensation is referred to as the transformation $[T_4]$. Let the relationship of the two submatrices in Eq. (12) be expressed by

$$[{}^{\alpha}_{\beta}] = [T_4][\alpha] \tag{15}$$

Substituting into Eq. (13) yields

$$U^* = \tfrac{1}{2}[\alpha]^T[T_4]^T[S][T_4][\alpha] = \tfrac{1}{2}[\alpha]^T[K^*][\alpha] \tag{16}$$

where $[K^*] = [S_{11} - S_{12}S_{22}^{-1}S_{21}]$ is the equivalent diagonal stiffness matrix and U^* is the strain energy expression for the equivalent continuum model that represents the original cell.

The coefficients of the various kinematic quantities in the strain energy expression are the equivalent continuum parameters. For example, the multiplier of the curvature term κ_2 is the equivalent EI_y for the repeated cell. In other words, the bending properties of the repeated structure around the y axis will be approximated by the behavior of a continuous beam with an EI_y of this value. This equivalent quantity can be used in the governing differential equation or in a finite element solution to analyze the equivalent continuum representation of the original discrete structure.

The kinematic quantities retained for an equivalent plate structure are ϵ_x, ϵ_y, γ_{xy}, $\epsilon_{x,z}$, $\epsilon_{y,z}$, $\gamma_{xy,z}$, γ_{xz}, and γ_{yz}. Numerical results for beam- and plate-like structures are presented in the next section.

VI. Numerical Examples

The procedure developed was validated by applying it to problems in Ref. 22 where an extensive comparison of the equivalent continuum results with the exact lattice structure is made. The beam-like lattice is shown in Fig. 1. This repeated element has 9 nodes each with 3 degrees of freedom for a total of 27 degrees of freedom. When the 6 rigid-body modes are discounted, the displaced configuration of the structure can be described with the 18 strain gradient terms as discussed in Sec. IV.

When the strain energy expression is reduced to the six independent kinematic variables defined in Eq. (11), the resulting coefficients of the kinematic variables for the three-dimensional beam are as shown in column 3 of Table 3. These results match the quantities presented in Ref. 22. In addition to the direct comparison with Ref. 22, these coefficients were checked by loading a cantilevered repeated element with an axial load, two pure moments, two stress loads, and a torsional load. The coefficients determined by these loading checks compared precisely with the computed coefficients. Thus, the procedure developed here checks with another analytic formulation and an empirical verification.

The influence of the secondary strain gradient terms [those terms denoted by $[\beta]$ in Eq. (13)] can be seen by comparing columns 2 and 3 of Table 3. As would be expected, the inclusion of the influence of these terms softens the stiffness coefficient. The reductions are in the range of 6.7-22.3%.

The plate-like lattice analyzed is the tetrahedral grid of Ref. 22. This structure has 6 nodes each with 3 degrees of freedom for a total of 18 degrees of freedom. The displaced configuration of the repeated element is described by 11 strain gradient terms. When the strain energy expression is reduced to the eight independent variables for the plate listed at the end of Sec. V, the results exactly match those found using the method presented in Ref. 22.

Table 3 Equivalent continuum parameters for the beam-like example

Kinematic variable	Coefficients before collapsing	Coefficients after collapsing[22]	Percent reduction
κ_1	4.589×10^6	3.563×10^6	−22.3
κ_2	7.687×10^7	7.170×10^7	−6.7
κ_3	7.687×10^7	7.170×10^7	−6.7
ϵ_x	2.217×10^7	1.721×10^7	−8.7
γ_{xy}	1.101×10^6	1.005×10^6	−8.7
γ_{xz}	1.101×10^6	1.005×10^6	−22.7

VII. Conclusions

A direct approach for creating an equivalent continuum representation for the stiffness properties of beam- and plate-like repeated latticed structures has been presented. The use of the displacement expansion in terms of strain gradient variables and the computer identification of dependency relations eliminates a possible source of errors in formulating equivalent continuum parameters. The analyst does not have to directly supply the secondary strain gradient terms that describe a structure. The possibility of not including an important term is greatly reduced and the possibility of including a redundant linearly dependent term is virtually eliminated.

One must input the stiffness matrix of the repeated element and choose the type of continuum representation desired. Any special kinematic constraints must also be supplied. The procedure is applicable to structures containing displacement and rotational degrees of freedom.

The procedure is based on formulating a strain energy for the discrete structure that is equivalent to the strain energy expression of the desired continuum. This is accomplished by expanding the displacement fields in terms of rigid-body and strain gradient quantities. A third-order representation of the displacement fields was used in this presentation, but higher-order representations are not difficult to formulate. Higher-order representations would be desirable for repeated elements with a large number of degrees of freedom.

The strain gradient quantities required are determined by using an elimination procedure that identifies linearly independent variables while computing the rank of the transformation matrix $[T_1]$. The strain energy expression is then reduced to a function of the desired kinematic quantities by a condensation procedure. The resulting coefficients of the kinematic variables are the parameters that define the equivalent continuum. In addition to allowing the response of the equivalent continuum to be determined, this procedure allows the physical characteristics of competing design configurations to be directly compared on a one-to-one basis by comparing the equivalent continuum parameters.

Acknowledgments

This study was developed as part of Research Contract SFRS F33615-82-K-3220 supported by AFWAL/FIBRA, Wright-Patterson Air Force Base, OH, conducted at the University of Colorado at Boulder.

References

[1]"Lattice Structures, State of the Art Report," *Journal of the Structural Div., ASCE*, Vol. 102, No. ST 11, Nov. 1976, pp. 2197-2230.

[2]Sherman, D. R., "Bibliography on Latticed Structures," *Journal of the Structural Div., ASCE*, Vol. 98, No. ST 7, July 1972, pp. 1545-1566.

[3]Nayfeh, A. H. and Hefzy, M. S., "Continuum Modeling of Three Dimensional Truss-like Space Structures," *AIAA Journal*, Vol. 16, Aug. 1978, pp. 779-787.

[4]Nayfeh, A. H. and Hefzy, M. S., "Continuum Modeling of the Mechanical and Thermal Behavior of Discrete Large Structures," *AIAA Journal*, Vol. 19, June 1981, pp. 766-773.

[5]Kollar, L., "Continuum Method of Analysis for Double-Layer Space Trusses with Upper and Lower Chord Planes of Different Rigidities," *Acta Technica*, Vol. 76, No. 1-2, 1974, pp. 53-63.

[6]Kollar, L., "Analysis of Double-Layer Space Trusses with Diagonally Square Mesh by the Continuum Method," *Acta Technica*, Vol. 76, No. 3-4, 1974, pp. 273-292.

[7]Kollar, L., "Continuum Method of Analysis for Double-Layer Space Trusses of Hexagonal over Triangular Mesh," *Acta Technica*, Vol. 86, No. 1-2, 1978, pp. 55-77.

[8]Sun, C. T. and Chen, C. C., "Transient Analysis of Large Frame Structures by Simple Models," Report on NSF Grant GI-41897.

[9]Sun, C. T. and Juang, J. N., "Parameter Estimation in Truss Beams Using Timoshenko Beam Model with Damping," Paper presented at Workshop on Applications of Distributed System Theory to the Control of the Large Space Structures, Jet Propulsion Laboratory, Pasadena, CA, July 1982.

[10]Sun, C. T., Lo, H., Chang, N. C., and Bogdanoff, J. L., "A Simple Continuum Model for Dynamic Analysis of Complex Plane Frame Structures," Report on NSF Grant GI-41897.

[11]Sun, C. T., Kim, B. J., and Bogdanoff, J. A., "On the Derivation of Equivalent Simple Models for Beams and Plate-like Structures in Dynamic Analysis," *Proceedings of 22nd AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference*, AIAA, NY, April 1981, pp. 523-532.

[12]Sun, C. T. and Yang, T. Y., "A Coupled Stress Theory for Gridwork-reinforced Media," *Journal of Elasticity*, Vol. 5, March 1975, pp. 45-58.

[13]Sun, C. T. and Yang, T. Y., "A Continuum Approach Toward Dynamics of Gridworks," *Journal of Applied Mechanics, Transactions of ASME*, Vol. 40, March 1973, pp. 186-192.

[14]Renton, J. D., "The Related Behavior of Plane Grids, Space Grids and Plates," *Space Structures*, Blackwell, Oxford, England, 1967, pp. 19-32.

[15]Renton, J. D., "General Properties of Space Grids," *International Journal of Mechanical Science*, Vol. 12, 1970, pp. 801-810.

[16]Schmidt, L. C. and Flower, W. R., "Analysis of Space Truss as Equivalent Plate," *Journal of the Structural Div., ASCE*, Vol. 97, No. ST 12, Dec. 1971, pp. 2777-2789.

[17]Schmidt, L. C., Morgan, P. R., and Hanaor, A., "Ultimate Load Testing of Space Trusses," *Journal of the Structural Div., ASCE*, Vol. 108, No. ST 6, June 1982, pp. 1324-1335.

[18]Schmidt, L. C., "Load Transfer by Shear and Torsion in Plate-like Space Trusses," *Civil Engineering Transaction, Institute of Engineering, Australia*, Vol. 22, 1980, pp. 97-103.

[19]Noor, A. K. and Nemeth, M. P., "Analysis of Spatial Beamlike Lattices with Rigid Joints," *Computer Methods in Applied Mechanics and Engineering*, Vol. 24, 1980, pp. 35-59.

[20]Noor, A. K. and Nemeth, M. P., "Micropolar Beam Models for Lattice Grids with Rigid Joints," *Computer Methods in Applied Mechanics and Engineering*, Vol. 21, 1980, pp. 249-263.

[21]Noor, A. K., "Thermal Stress Analysis of Double-Layer Grids," *Journal of The Structural Div., ASCE*, Vol. 104, No. ST 2, Feb. 1978, pp. 251-262.

[22]Noor, A. K., Anderson, M. S., and Green, W. H., "Continuum Models for Beam- and Plate-like Lattice Structures," *AIAA Journal*, Vol. 16, Dec. 1978, pp. 1219-1228.

[23]Noor, A. K. and Anderson, C. M., "Analysis of Beam-like Lattice Trusses," *Computer Methods in Applied Mechanics and Engineering*, Vol. 20, 1979, pp. 53-70.

[24]Noor, A. K., Green, W. H., and Anderson, M. S., "Continuum Models for Static and Dynamic Analysis of Repetitive Trusses," *Proceedings of 18th AIAA/ASME/SAE Structures, Structural Dynamics and Materials Conference*, AIAA, NY, March 1977, pp. 299-310.